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Almost convergence and a core theorem for double sequences

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Abstract

The idea of almost convergence was introduced by Moricz and Rhoades [Math. Proc. Cambridge Philos. Soc. 104 (1988) 283–294] and they also characterized the four dimensional strong regular matrices. In this paper we define the M-core for double sequences and determine those four dimensional matrices which transform every bounded double sequence $x = [x_{jk}]$ into one whose core is a subset of the M-core of x.

Keywords: Double sequences; Almost convergence; Strongly regular matrices; Core of a sequence

1. Introduction

A double sequence $x = [x_{jk}]_{j,k=0}^{\infty}$ is said to be *convergent in the Pringsheim sense* or P-convergent if for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $|x_{jk} - L| < \varepsilon$ whenever $j, k > N$ and L is called the *Pringsheim limit* (denoted by P-lim $x = L$) (cf. [8]). We will denote the space of P-convergent sequences by c_2.

A double sequence x is *bounded* if there exists a positive number M such that $|x_{jk}| < M$ for all j and k, i.e., if

$$\|x\| = \sup_{j,k} |x_{jk}| < \infty.$$

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Note that in contrast to the case for single sequences, a convergent double sequence need not be bounded.

Let $A = [a_{jk}^{mn}]_{j,k=0}^{\infty}$ be a doubly infinite matrix of real numbers for all $m, n = 0, 1, \ldots$. Forming the sums

$$y_{mn} = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} a_{jk}^{mn} x_{jk},$$

called the *A-means* of the double sequence x, yields a method of summability. We say that a sequence x is *A-summable* to the limit s if the A-means exist for all $m, n = 0, 1, \ldots$ in the sense of Pringsheim's convergence,

$$\lim_{p,q\to\infty} \sum_{j=0}^{p} \sum_{k=0}^{q} a_{jk}^{mn} x_{jk} = y_{mn}$$

and

$$\lim_{m,n\to\infty} y_{mn} = s.$$

A two dimensional matrix transformation is said to be *regular* if it maps every convergent sequence into a convergent sequence with the same limit. In 1926 Robinson [9] presented a four dimensional analogue of regularity for double sequences in which he added an additional assumption of boundedness: A four dimensional matrix A is said to be *bounded-regular* or *RH-regular* if it maps every bounded P-convergent sequence into a P-convergent sequence with the same P-limit.

The following is a four dimensional analogue of the well-known Silverman–Toeplitz theorem [1].

Theorem 1.1. *The four dimensional matrix A is bounded-regular or RH-regular if and only if* (*see Hamilton* [2], *Robinson* [9])

(RH_1) $\text{P-}\lim_{m,n} a_{jk}^{mn} = 0$ $(j, k = 0, 1, \ldots)$,
(RH_2) $\text{P-}\lim_{m,n} \sum_{j,k=0,0}^{\infty,\infty} a_{jk}^{mn} = 1$,
(RH_3) $\text{P-}\lim_{m,n} \sum_{j=0}^{\infty} |a_{jk}^{mn}| = 0$ $(k = 0, 1, \ldots)$,
(RH_4) $\text{P-}\lim_{m,n} \sum_{k=0}^{\infty} |a_{jk}^{mn}| = 0$ $(j = 0, 1, \ldots)$,
(RH_5) $\sum_{j,k=0,0}^{\infty,\infty} |a_{jk}^{mn}| \leqslant C < \infty$ $(m, n = 0, 1, \ldots)$.

Note that (RH_1) is a consequence of each of (RH_3) and (RH_4).

The *core* (or K-core) of a real number sequence is defined to be the closed interval $[\liminf x, \limsup x]$. The well-known Knopp core theorem states as follows (see Knopp [3], Maddox [5]).

Theorem 1.2. *In order that $L(Ax) \leqslant L(x)$ for every bounded sequence $x = (x_k)$, it is necessary and sufficient that $A = (a_{nk})$ should be regular and $\lim_n \sum_{k=0}^{\infty} |a_{nk}| = 1$, where $L(x) = \limsup x$.*

Recently in [7], Patterson extended this idea for double sequences by defining the Pringsheim core as follows.

Let P-$C_n\{x\}$ be the least closed convex set that includes all points x_{jk} for $j, k > n$; then the *Pringsheim core* of the double sequence $x = [x_{jk}]$ is the set P-$C\{x\} = \bigcap_{n=1}^{\infty}[\text{P-}C_n\{x\}]$.

Note that the Pringsheim core of a real-valued bounded double sequence is the closed interval [P-lim inf x, P-lim sup x].

In this regard, Patterson proved the following

Theorem 1.3. *If A is a four dimensional matrix, then for all real-valued double sequences x,*

$$\text{P-}\limsup Ax \leqslant \text{P-}\limsup x$$

if and only if

(1) *A is RH-regular and*
(2) P-$\lim_{mn} \sum_{j,k=0,0}^{\infty,\infty} |a_{jk}^{mn}| = 1$.

In the present paper we define the M-core of a double sequence by using the idea of almost convergence introduced and studied by Moricz and Rhoades [6], and then proved an analogue of Theorem 1.3.

2. Almost convergence and M-core

The notion of almost convergence for single sequences was introduced by Lorentz [4]. Recently Moricz and Rhoades [6] extended this idea for double sequences.

A double sequence $x = [x_{jk}]_{j,k=0}^{\infty}$ of real numbers is said to be *almost convergent* to a limit s if

$$\lim_{p,q\to\infty} \sup_{m,n\geqslant 0} \left| \frac{1}{pq} \sum_{j=m}^{m+p-1} \sum_{k=n}^{n+q-1} x_{jk} - s \right| = 0,$$

that is, the average value of $[x_{jk}]$ taken over any rectangle $\{(j,k)\colon\ m \leqslant j \leqslant m+p-1;\ n \leqslant k \leqslant n+q-1\}$ tends to s as both p and q tend to ∞, and this convergence is uniform in m and n.

Note that a convergent single sequence is also almost convergent but for a double sequence this is not the case, that is, a convergent double sequence need not be almost convergent. However every bounded convergent double sequence is almost convergent.

Using the idea of almost convergence, Lorentz [4] introduced and characterized strongly regular matrices. We say that a matrix A is *strongly regular* if every almost convergent sequence x is A-summable to the same limit, and the A-means are also bounded.

If a double sequence x is almost convergent to s, then we write f_2-$\lim x = s$ and f_2 for the space of almost convergent double sequences.

In [6], Moricz and Rhoades gave four dimensional analogue of strongly regular matrices as follows.

Theorem 2.1. *Necessary and sufficient conditions for a matrix $A = [a_{jk}^{mn}]$ to be strongly regular are that A is bounded-regular and satisfies the following two conditions*:

(MR$_1$) $\lim_{m,n\to\infty} \sum_{j,k=0,0}^{\infty,\infty} |\Delta_{10} a_{jk}^{mn}| = 0$,
(MR$_2$) $\lim_{m,n\to\infty} \sum_{j,k=0,0}^{\infty,\infty} |\Delta_{01} a_{jk}^{mn}| = 0$,

where $\Delta_{10} a_{jk}^{mn} = a_{jk}^{mn} - a_{j+1,k}^{mn}$ *and* $\Delta_{01} a_{jk}^{mn} = a_{jk}^{mn} - a_{j,k+1}^{mn}$ $(j,k = 0,1,\ldots)$.

We define the following. Let us write

$$L^\star(x) = \limsup_{p,q\to\infty} \sup_{m,n\geqslant 0} \frac{1}{pq} \sum_{j=m}^{m+p-1} \sum_{k=n}^{n+q-1} x_{jk}.$$

Then we define the M-core of a real-valued bounded double sequence x to be the closed interval $[-L^\star(-x), L^\star(x)]$.

Since every bounded convergent double sequence is almost convergent, we have

$$L^\star(x) \leqslant \text{P-}\limsup x,$$

and hence it follows that M-core$\{x\} \subseteq$ P-core$\{x\}$ for a bounded double sequence $x = [x_{jk}]_{j,k=0}^{\infty}$.

We quote here the following useful lemma.

Lemma 2.2 [7]. *If A is a real or complex-valued four dimensional matrix such that* (RH$_3$), (RH$_4$), *and*

$$\text{P-}\limsup_{m,n} \sum_{j,k=0,0}^{\infty,\infty} \left|a_{jk}^{mn}\right| = M$$

hold, then for any bounded double sequence x we have

$$\text{P-}\limsup |Ax| \leqslant M\big(\text{P-}\limsup |x|\big).$$

3. Main result

Here we prove a core theorem for double sequences making use of four dimensional strongly regular matrices due to Moricz and Rhoades [6].

Theorem 3.1. *For every bounded double sequence x*,

$$L(Ax) \leqslant L^\star(x) \tag{3.1}$$

(*or* P-core$\{Ax\} \subseteq$ M-core$\{x\}$) *if and only if*

(i) $A = [a_{jk}^{mn}]$ *is strongly regular and*
(ii) $\text{P-}\lim_{m,n\to\infty} \sum_{j,k=0,0}^{\infty,\infty} |a_{jk}^{mn}| = 1$.

Proof. *Necessity*. Let us consider a bounded double sequence x to be almost convergent to s. Then we have $L^\star(x) = -L^\star(-x)$. By (3.1), we get

$$s = -L^\star(-x) \leqslant -L(-Ax) \leqslant L(Ax) \leqslant L^\star(x) = s.$$

Hence Ax is P-convergent and P-lim $Ax = f_2$- $\lim x = s$, and so A is strongly regular, i.e., condition (i) holds.

Since every strongly regular matrix is also bounded-regular, by Lemma 2.2 there exists a bounded double sequence x such that $\limsup |x| = 1$ and P-lim sup $Ax = C$, where C is defined by (RH_5). Therefore we have

$$1 \leqslant \text{P-lim}\inf_{m,n} \sum_{j,k=0,0}^{\infty,\infty} \left|a_{jk}^{mn}\right| \leqslant \text{P-lim}\sup_{m,n} \sum_{j,k=0,0}^{\infty,\infty} \left|a_{jk}^{mn}\right| \leqslant 1,$$

i.e., condition (ii) holds.

Sufficiency. Given $\varepsilon > 0$, we can find fixed integers $p, q \geqslant 2$ such that

$$\frac{1}{pq} \sum_{j=m}^{m+p-1} \sum_{k=n}^{n+q-1} x_{jk} < L^\star(x) + \varepsilon. \tag{3.2}$$

Now as in [6], we can write

$$y_{MN} = \sum_{j,k=0,0}^{\infty,\infty} a_{jk}^{MN} x_{jk} = \Sigma_1 + \Sigma_2 + \Sigma_3 + \Sigma_4 + \Sigma_5 + \Sigma_6 + \Sigma_7 + \Sigma_8, \tag{3.3}$$

where

$$\Sigma_1 = \frac{1}{pq} \sum_{m,n=0,0}^{\infty,\infty} a_{mn}^{MN} \sum_{j=m}^{m+p-1} \sum_{k=n}^{n+q-1} x_{jk},$$

$$\Sigma_2 = \frac{-1}{pq} \sum_{j=0}^{p-2} \sum_{k=0}^{q-2} x_{jk} \sum_{m=0}^{j} \sum_{n=0}^{k} a_{mn}^{MN},$$

$$\Sigma_3 = -\frac{1}{pq} \sum_{j=p-1}^{\infty} \sum_{k=0}^{q-2} x_{jk} \sum_{m=j-p+1}^{j} \sum_{n=0}^{k} a_{mn}^{MN},$$

$$\Sigma_4 = -\frac{1}{pq} \sum_{j=0}^{p-2} \sum_{k=q-1}^{\infty} x_{jk} \sum_{m=0}^{j} \sum_{n=k-q+1}^{k} a_{mn}^{MN},$$

$$\Sigma_5 = -\sum_{j=p-1}^{\infty} \sum_{k=q-1}^{\infty} x_{jk} \left\{ \frac{1}{pq} \sum_{m=j-p+1}^{j} \sum_{n=k-q+1}^{k} a_{mn}^{MN} - a_{jk}^{MN} \right\},$$

$$\Sigma_6 = \sum_{j=0}^{p-2} \sum_{k=0}^{q-2} a_{jk}^{MN} x_{jk},$$

$$\Sigma_7 = \sum_{j=p-1}^{\infty} \sum_{k=0}^{q-2} a_{jk}^{MN} x_{jk},$$

$$\Sigma_8 = -\sum_{j=0}^{p-2}\sum_{k=q-1}^{\infty} a_{jk}^{MN} x_{jk}.$$

Using the conditions of strong regularity of A, we observe that

$$|\Sigma_2| \leqslant \|x\| \sum_{m=0}^{p-2}\sum_{n=0}^{q-2}\left|a_{mn}^{MN}\right| \to 0 \quad (M, N \to \infty),$$

and

$$|\Sigma_6| \leqslant \|x\| \sum_{j=0}^{p-2}\sum_{k=0}^{q-2}\left|a_{jk}^{MN}\right| \to 0 \quad \text{by } (\mathrm{RH}_1),$$

$$|\Sigma_3| \leqslant \|x\| \sum_{m=0}^{\infty}\sum_{n=0}^{q-2}\left|a_{mn}^{MN}\right| \to 0,$$

and

$$|\Sigma_7| \leqslant \|x\| \sum_{j=p-1}^{\infty}\sum_{k=0}^{q-2}\left|a_{jk}^{MN}\right| \to 0 \quad \text{by } (\mathrm{RH}_3),$$

$$|\Sigma_4| \to 0 \quad \text{and} \quad |\Sigma_8| \to 0 \quad \text{by } (\mathrm{RH}_4).$$

Now

$$|\Sigma_5| \leqslant \frac{\|x\|}{pq}\sum_{r=0}^{p-1}\sum_{s=0}^{q-1}\left\{(p-r-1)\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\left|\Delta_{10}a_{jk}^{MN}\right| + (q-s-1)\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\left|\Delta_{01}a_{jk}^{MN}\right|\right\} \to 0 \quad \text{by } (\mathrm{MR}_1) \text{ and } (\mathrm{MR}_2).$$

Therefore we have by (3.3),

$$\begin{aligned} L(Ax) &\leqslant \limsup_{M,N} \sum_{m,n=0,0}^{\infty,\infty} a_{mn}^{MN} \frac{1}{pq}\sum_{j=m}^{m+p-1}\sum_{k=n}^{n+q-1} x_{jk} \\ &\leqslant \limsup_{M,N} \left|\sum_{m,n=0,0}^{\infty,\infty}\left(\frac{|a_{mn}^{MN}| + a_{mn}^{MN}}{2} + \frac{|a_{mn}^{MN}| - a_{mn}^{MN}}{2}\right)\frac{1}{pq}\sum_{j=m}^{m+p-1}\sum_{k=n}^{n+q-1} x_{jk}\right| \\ &\leqslant \limsup_{M,N}\left\{\sum_{m,n=0,0}^{\infty,\infty}\left|a_{mn}^{MN}\right|\left|\frac{1}{pq}\sum_{j=m}^{m+p-1}\sum_{k=n}^{n+q-1} x_{jk}\right| + \|x\|\sum_{m,n=0,0}^{\infty,\infty}\left(\left|a_{mn}^{MN}\right| - a_{mn}^{MN}\right)\right\}. \end{aligned}$$

Now conditions (RH_1), (RH_5) and (ii) yield

$$L(Ax) \leqslant L^{\star}(x) + \varepsilon.$$

Since ε is arbitrary we finally have

$$L(Ax) \leqslant L^\star(x).$$

This completes the proof of the theorem. □

4. Examples

4.1. Almost convergent sequences

(i) Define the double sequence $x = [x_{jk}]$ by

$$x_{jk} = \begin{cases} 1 & \text{if } j \text{ is odd, for all } k, \\ 0 & \text{otherwise.} \end{cases}$$

Then x is almost convergent to $1/2$.

(ii) Define $x = [x_{jk}]$ by

$$x_{jk} = (-1)^j \quad \text{for all } k.$$

Then x is almost convergent to 0.

4.2. Strongly regular matrix

Define $A = [a_{jk}^{mn}]$ by

$$a_{jk}^{mn} = \begin{cases} \frac{1}{m^2} & \text{if } m = n \text{ and } j, k \leqslant m \text{ (even)}, \\ \frac{1}{m^2 - m} & \text{if } m = n,\ j \neq k \text{ and } j, k \leqslant m \text{ (odd)}, \\ 0 & \text{otherwise.} \end{cases}$$

We can easily verify that A is strongly regular, that is, conditions (RH_1)–(RH_5), (MR_1) and (MR_2) hold. Moreover, for the sequence (i), we have

$$\begin{aligned} \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} a_{jk}^{mn} x_{jk} &= a_{11}^{mm} x_{11} + a_{12}^{mm} x_{12} + \cdots + a_{1m}^{mm} x_{1m} \\ &\quad + a_{21}^{mm} x_{21} + a_{22}^{mm} x_{22} + \cdots + a_{2m}^{mm} x_{2m} \\ &\quad + a_{31}^{mm} x_{31} + a_{32}^{mm} x_{32} + a_{33}^{mm} x_{33} + \cdots + a_{3m}^{mm} x_{3m} \\ &\quad\ \ \vdots \\ &\quad + a_{m-1,1}^{mm} x_{m-1,1} + \cdots + a_{m-1,m}^{mm} x_{m-1,m} \\ &\quad + a_{m1}^{mm} x_{m1} + \cdots + a_{mm}^{mm} x_{mm} \\ &= \frac{m}{m^2} \cdot \frac{m}{2} \quad \text{(if } m \text{ is even)} \\ &\to \frac{1}{2} \quad \text{as } m, n \to \infty. \end{aligned}$$

Similarly

$$\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} a_{jk}^{mn} x_{jk} = \frac{m-1}{m^2-m} \cdot \frac{m+1}{2} \quad \text{(if } m \text{ is odd)}$$
$$\to \frac{1}{2} \quad \text{as } m, n \to \infty.$$

That is

$$\text{P-lim}\, Ax = \frac{1}{2} = f_2\text{-}\lim x,$$

and so A transforms almost convergent sequence into convergent (P-convergent) to the same limit.

4.3. Bounded-regular matrix which is not strongly regular

In Section 4.2, A is strongly regular and so bounded regular. Let us define $A = [a_{jk}^{mn}]$ as

$$a_{jk}^{mn} = \begin{cases} \frac{2}{m^2} & \text{if } m = n,\ j+k = \text{even, and } j, k \leqslant m \text{ (even)}, \\ \frac{1}{m^2-m} & \text{if } m = n,\ j \neq k \text{ and } j, k \leqslant m \text{ (odd)}, \\ 0 & \text{otherwise}. \end{cases}$$

Then A is bounded-regular but not strongly regular. Conditions (RH_1)–(RH_5) can easily be verified. But

$$\lim_{m,n} \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} \left|a_{jk}^{mn} - a_{j+1,k}^{mn}\right| = \begin{cases} 2 & \text{if } m \text{ is even}, \\ 0 & \text{if } m \text{ is odd}, \end{cases}$$

and also

$$\lim_{m,n} \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} \left|a_{jk}^{mn} - a_{j,k+1}^{mn}\right| = \begin{cases} 2 & \text{if } m \text{ is even}, \\ 0 & \text{if } m \text{ is odd}. \end{cases}$$

Therefore conditions (MR_1) and (MR_2) do not hold and so A is not strongly regular.

4.4. In Theorem 3.1, strong regularity of A cannot be replaced by bounded-regularity.

Consider the matrix $A = [a_{jk}^{mn}]$ as defined in Section 4.3. This is bounded-regular but not strongly regular, and also

$$\text{P-lim}_{m,n} \sum_{j,k=0,0}^{\infty,\infty} \left|a_{jk}^{mn}\right| = 1,$$

i.e., condition (ii) of Theorem 3.1 holds. Take the bounded double sequence $x = [x_{jk}]$ defined by $x_{jk} = (-1)^{j+k}$, which is almost convergent to zero, that is,

$$L^\star(x) = 0.$$

Now

$$\sum_{j,k} a_{jk}^{mn} x_{jk} = \begin{cases} \frac{2}{m^2} \cdot \frac{m}{2} \cdot m & \text{if } m \text{ is even,} \\ \frac{-1}{m^2-m} \cdot m & \text{if } m \text{ is odd.} \end{cases}$$

Therefore

$$\limsup_{m,n} \sum_{j,k} a_{jk}^{mn} x_{jk} = 1$$

and

$$\liminf_{m,n} \sum_{j,k} a_{jk}^{mn} x_{jk} = 0,$$

i.e., $L(Ax) = 1$. Hence $L(Ax) > L^\star(x)$, that is (3.1) does not hold.

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